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Realisations of the Representations of para-Fermi Algebra—Part III: Fock Space of Fermi Operators

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Abstract

Following the idea given by Kademova (1969b), Green's isomorphisms of the para-Fermi algebra F^1_{2n} into the space of all bilinear products of para-Fermi type operators are constructed. The induced transformations in Fock space of Fermi operators are considered. The difference between the induced transformations in Fock space of Bose operators and in Fock space of Fermi operators is discussed.

1. *Introduction*

A constructive method for finding the generators of para-Fermi algebra with two generators only as functions of the creation and annihilation operators of para-Bose type and for realising all the representations of the para-Fermi algebra in Fock space of two Bose operators has been given in the paper by Kademova (1969b). An extension of the method for realisation of the representations of the para-Fermi algebra with $2n$ generators in Fock space of 2^n Bose operators has been made in the paper by Kademova & Kálnay (1969).

In the present paper we construct Green's isomorphic mapping of the para-Fermi algebra into the space of the bilinear products of creation and annihilation para-Fermi operators of arbitrary order of parastatistics p (Section 2).

In the Fock space of Fermi operators the transformations induced by the isomorphic algebra are constructed. The space is reduced into invariant subspaces of a fixed number of particles. The transformations induced in the single-particle subspace form para-Fermi algebra of parastatistics $p = 1$ (Section 3).

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The difference between the induced transformations in Fock space of Bose operators and in Fock space of Fermi operators is discussed (Section 4).

Throughout this paper we use the notations introduced by Kademova (1969b) and Kademova & Kálnay (1969), except when otherwise stated.

2. *Green's Isomorphism of the para-Fermi Algebra into the Space of Bilinear Products of Para-Fermi Operators*

As in Kademova (1969b), using the results of the realisations of Lie algebras by means of parafield operators (Kademova, 1969a) we can prove the following theorem:

Theorem:

For arbitrary para-Fermi algebra $\mathbf{F}_{2n}^{p}$ the mapping

$$i_q : (\mathscr{F}_k)_q{}^p = \sum_{i,j=1}^{2p.n} (F_k{}^p)_{ij} \tfrac{1}{2}[\overset{+}{f_i^q}, f_j{}^q]_-, \qquad k = 1, 2, \ldots, n \tag{2.1}$$

is a Green isomorphism of $\mathbf{F}_{2n}^{p}$ into $\boldsymbol{\epsilon}_{2p.n+1}^{q(2)}$.†

In the case of $p = q = 1$ we get‡

$$i_1 : (\mathscr{F}_k)_1{}^1 = \sum_{i,j=1}^{2^n} (F_k{}^1)_{ij} \overset{+}{f_i} f_j \tag{2.2}$$

Using the matrix representation of $\mathbf{F}_{2n}^{1}$ generators (Kademova & Kálnay, 1969, formula (2.1) we write the generators of the algebra $\mathscr{F}$ in the form:

$$\begin{aligned} \overset{+}{\mathscr{F}}_i &= \sum_{\mu_1,\ldots,\mu_n=0}^{1} (-1)^{\sum_{k=i+1}^{n} \mu_k} \theta(1-\mu_i) . f_{\sum_{k=1}^{n} \mu_k 2^{k-1}+2^{i-1}+1} f_{\sum_{k=1}^{n} \mu_k+2^{i-1}} \\ \mathscr{F}_i &= (\overset{+}{\mathscr{F}}_i)^+ \end{aligned} \tag{2.3}$$

Let us remember that for the objects $\overset{+}{\mathscr{F}}_i$ and $\mathscr{F}_i$ only the Green product s defined.

3. *Transformations Induced by $\mathscr{F}$ in the Fock Space of the Algebra* $\mathbf{F}_{2^n+1}^{1}$

The Fock space of the algebra $\mathbf{F}_{2^n+1}^{1}$ is spanned on the monomials

$$|\alpha_1, \alpha_2, \ldots, \alpha_{2^n}\rangle = \prod_{i=1}^{2^n} (\overset{+}{f_i})^{\alpha_i} |0\rangle \tag{3.1}$$

where $\alpha_1, \ldots, \alpha_{2^n} = 0, 1$.

† $\boldsymbol{\epsilon}_{2p.n+1}^{q(2)}$ is introduced for the operators $\overset{+}{f_i^q}$, $f_i{}^q$ in a way similar to that in Kademova (1969b) for the operators $\overset{+}{b_i^q}$, $b_i{}^q$.

‡ For simplicity we shall from now on write $\mathscr{F}_k$ instead of $(\mathscr{F}_k)_1{}^1$ and $\overset{+}{f_i}$, f_i instead of $\overset{+}{f_i^1}$, $f_i{}^1$.

As in Kademova & Kálnay (1969) using the formulas (2.3) and (3.1) we write the induced transformations in the form:

$$\overset{+}{\mathscr{F}}_i|\alpha_1,\ldots,\alpha_{2^n}\rangle = \sum_{\mu_1,\ldots,\mu_n=0}^{1} (-1)^{\sum_{k=i+1}^{n}\mu_k}\,\theta(1-\mu_i)\,.$$
$$.\,\theta\Big(\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+1}\Big)\,\theta\Big(1-\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+2^{i-1}+1}\Big)$$
$$.\,|\alpha_1,\ldots,\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+1}-1,\ldots,$$
$$\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+2^{i-1}+1}+1,\ldots,\alpha_{2^n}\rangle$$

$$\mathscr{F}_i|\alpha_1,\ldots,\alpha_{2^n}\rangle = \sum_{\mu_1,\ldots,\mu_n=0}^{1} (-1)^{\sum_{k=i+1}^{n}\mu_k}\,\theta(1-\mu_i) \tag{3.2}$$
$$.\,\theta\Big(\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+2^{i-1}+1}\Big)\,\theta\Big(1-\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+1}\Big)$$
$$.\,|\alpha_1,\ldots,\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+1}+1,\ldots,$$
$$\alpha_{\sum_{k=1}^{n}\mu_k 2^{k-1}+2^{i-1}+1}-1,\ldots,\alpha_{2^n}\rangle$$

The single-particle subspace is spanned on the monomials

$$|\alpha_1,\ldots,\alpha_{2^n}\rangle = \prod_{i,\epsilon} (\overset{+}{\mathscr{F}}_i)^{\alpha_{i\epsilon}}|1,0,\ldots,0\rangle \tag{3.3}$$

with the restriction

$$\sum_{i=1}^{2^n} \alpha_i = 1$$

Straightforward calculations show that in this subspace the induced transformations form para-Fermi algebra with $2n$ generators of para-statistics $p = 1$ (i.e. Fermi algebra).

The subspaces for which

$$\sum_{i=1}^{2^n} \alpha_i = \text{const} > 1$$

are also invariant under the induced transformations. But the transformations do not form a representation of the para-Fermi algebra with $2n$ generators. These subspaces are generally reducible with respect to the transformations induced by $\overset{+}{\mathscr{F}}_i$, $\mathscr{F}_i$ for each fixed i $(i = 1,2,\ldots,n)$.

The meaning of this statement is the following. Let us take the vector $|\alpha^0\rangle = |\alpha_1{}^0,\ldots,\alpha_{2^n}^0\rangle$. Using formula (3.2) the following vectors can be constructed for each fixed i:

$$|\alpha_h{}^i\rangle = (\overset{+}{\mathscr{F}}_i)^{\gamma_i}|\alpha^0\rangle \tag{3.4}$$
$$|\alpha_l{}^i\rangle = (\mathscr{F}_i)^{\beta_i}|\alpha^0\rangle$$

The vectors (3.4) satisfy the conditions:

$$\overset{+}{\mathscr{F}}_i|\alpha_h{}^i\rangle = 0$$
$$\mathscr{F}_i|\alpha_l{}^i\rangle = 0$$

γ_i and β_i depend on the choice of the generators $\overset{+}{\mathscr{F}}_i$, $\mathscr{F}_i$ and the vector $|\alpha^0\rangle$. The vectors $|\alpha_l{}^i\rangle$ and $|\alpha_h{}^i\rangle$ are generally linear combinations of the vectors (3.1). In the subspace spanned on the vectors $(\overset{+}{\mathscr{F}})^{\delta_i}|\alpha_l{}^i\rangle$ $(0 \leqslant \delta_i \leqslant \beta_i + \gamma_i)$ the operators $\overset{+}{\mathscr{F}}_i$ and $\mathscr{F}_i$ form para-Fermi algebra with two generators of parastatistics $p_i = \beta_i + \gamma_i$. If we denote by N the number of the non-zero indices $\alpha_k{}^0$, $k = 1,\ldots,2^n$, then

$$p_i \leqslant \begin{cases} N, & N \leqslant 2^{n-1} \\ 2^n - N & N \geqslant 2^{n-1} \end{cases} \tag{3.5}$$

4. *Discussion*

In this section we compare the properties of the transformations induced by the para-Fermi algebra in Fock space of Bose operators with those induced in Fock space of Fermi operators.

In Kademova (1969b) and Kademova & Kálnay (1969), using the matrix representation of $2n$ Fermi operators, a para-Fermi algebra with $2n$ generators has been constructed by means of 2^n Bose operators. This algebra induces transformations in the Fock space of 2^n Bose operators which leave invariant the subspaces containing a fixed number of Bose particles. These transformations form para-Fermi algebra with $2n$ generators of order of parastatistics $p = m$ in each subspace of m particles ($m = 1, 2,\ldots$).

In the previous sections, in a similar way, we have constructed a para-Fermi algebra with $2n$ generators by means of 2^n Fermi operators. In the Fock space of 2^n Fermi operators the induced transformations also leave invariant the subspaces of a fixed number of particles. But only in the single-particle subspace do they form a para-Fermi algebra with $2n$ generators of parastatistics $p = 1$ (i.e. Fermi algebra). In the other subspaces the induced transformations do not generally form a representation of the para-Fermi algebra with $2n$ generators.

To illustrate this let us consider the case of $n = 2$ and $m = 2$. The generators of the isomorphic algebra are:

$$\mathscr{F}_1 = \overset{+}{f}_1 f_2 - \overset{+}{f}_3 f_4$$
$$\mathscr{F}_2 = \overset{+}{f}_1 f_3 + \overset{+}{f}_2 f_4$$
$$\overset{+}{\mathscr{F}}_1 = (\mathscr{F}_1)^+$$
$$\overset{+}{\mathscr{F}}_2 = (\mathscr{F}_2)^+$$

The subspace is six dimensional. It can readily be checked that it is reducible into two three-dimensional subspaces:

$$\overset{+}{f}_1 \overset{+}{f}_3|0\rangle, \qquad (\overset{+}{f}_2 \overset{+}{f}_3 - \overset{+}{f}_1 \overset{+}{f}_4)|0\rangle, \qquad -2\overset{+}{f}_2 \overset{+}{f}_4|0\rangle$$

and

$$\overset{+}{f}_1 \overset{+}{f}_2|0\rangle, \qquad (\overset{+}{f}_2 \overset{+}{f}_3 + \overset{+}{f}_1 \overset{+}{f}_4)|0\rangle, \qquad 2\overset{+}{f}_3 \overset{+}{f}_4|0\rangle$$

the first being invariant under the transformations induced by the generators $\overset{+}{\mathscr{F}}_1$, $\mathscr{F}_1$ and the second under those induced by $\overset{+}{\mathscr{F}}_2$, $\mathscr{F}_2$. In each of these subspaces the induced transformations form a para-Fermi algebra with two generators of parastatistics $p = 2$. In the whole six-dimensional space the induced transformations do not form a representation of the para-Fermi algebra with four generators. They form only a representation of the direct product of two para-Fermi algebras with two generators each.

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